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Single and Multiple point processes:

TIH, hazard, survival, autocorrelation, cross correlation

Single spike trains basic terms:

- ISI- inter-spike interval, the time between two adjacent spikes
- TIH- time interval histogram
- Survivor function- the probability for at least one spike until a certain time point t:

$$survivor(t) = 1 - CDF\big(ISI(t)\big) = 1 - \sum_{i=1}^{t} PDF(ISI = i)$$

- Hazard function- the probability for spike at a certain time point t given that no spike occurred before:

$$hazard(t) = \frac{PDF(ISI = t)}{survivor(t)}$$

- Autocorrelation- the number of spikes or probability for spike or the rate of spikes at a certain time (t) after a spike (not necessarily the preceding one).

When are we interested in those functions?

- Survival analysis function (survival and hazards) are commonly used to quantify effects of anticipation and pattern learning.
- Autocorrelation, as generalized method to imply on relations between spikes, can be used for detecting patterns of modulations in spiking activity, for example oscillations and decay time.

Class Exercise: calculating the survival analysis

Given a neuron with equal probabilities of ISI of 5,10,15 sec, plot the:

- TIH
- CDF
- Survivor function
- Hazard function

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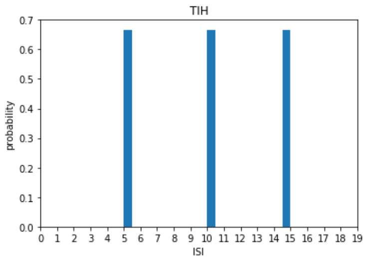
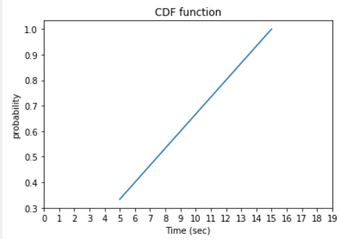
Solution:

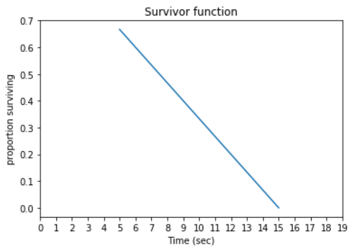
```
import numpy as np
import matplotlib.pyplot as plt

isiVec=np.array([5,10,15])
pdfVec=np.array([1/3,1/3,1/3])

#TIH
plt.figure(1)
plt.hist(isiVec, bins=20, normed=1)
plt.xticks(range(20))
plt.ylabel('probability')
plt.xlabel('ISI')
plt.title('TIH')
```



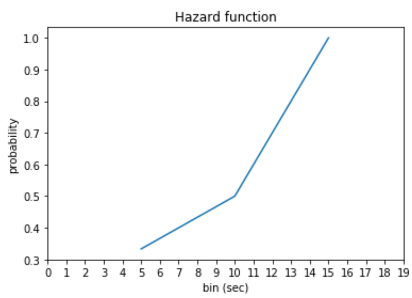

```
#CDF function
cdfVec=np.array([1/3,2/3,1])
plt.figure(2)
plt.plot(isiVec,cdfVec)
plt.xticks(range(20))
plt.ylabel('probability')
plt.xlabel('Time (sec)')
plt.title('CDF function')
```




```
#survivor function
survivorVec=1-cdfVec
plt.figure(3)
plt.plot(isiVec,survivorVec)
plt.xticks(range(20))
plt.ylabel('proportion surviving')
plt.xlabel('Time (sec)')
plt.title('Survivor function')
```




```
#hazard function
survivorVec4Hazard=[1,2/3,1/3]
hazardVec=np.divide(pdfVec,survivorVec4Hazard)
plt.figure(4)
plt.plot(isiVec,hazardVec)
plt.xticks(range(20))
plt.ylabel('probability')
plt.xlabel('bin (sec)')
plt.title('Hazard function')
```



The Hazard function properties:

Definitions:

$$\textbf{Continuous:}\quad hazard(t) = \frac{PDF(ISI=t)}{survivor(t)}$$

$$\textbf{Discrete:}\ hazard(t) = \frac{PDF(ISI=t)}{survivor(t-\epsilon)} \qquad (\epsilon = 1\ bin)$$

- Represents Bayesian probability- the probability for a spike at time (t) *given* that no spike occurred before
- Non-negative, $h(t) \geq 0$
- Not a probability function and therefore $\int h(t)dt \to \infty$
- In Poisson distribution:
    - $h(t)$ is constant, representing the process is renewal
    - $h(t) = r$
    - Noisier within longer ISI (due to final sampling)

Class Exercise: Discrete Poisson distribution

A neuron has a firing probability of $P(t)$ only at the time points from the given series $t = nT$:

$$P(t) = \begin{cases} p & t = nT \\ 0 & else \end{cases}$$

Plot the survivor function and the hazard.

Solution:

To calculate the hazard function, we need to calculate the probabilities for given ISI. Given bin sizes with the width T, for any $ISI = nT$:

$$p(ISI = nT) = (1-p)^{n-1} \cdot p$$

Therefore, the CDF(ISI) is given by:

$$CDF(nT) = \sum_{i=1}^{n} p(ISI = nT) = p \cdot (1-p)^{1-1} + p \cdot (1-p)^{2-1} + p \cdot (1-p)^{3-1} + \cdots$$

$$= p \cdot 1 + p \cdot (1-p)^{1} + p \cdot (1-p)^{2} + \cdots$$

Based on the fact the $p \leq 1$, We'll assume $p < 1$ and use the geometric convergence:

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \sum_{k=0}^{n-1} ar^k = a \cdot \frac{1 - r^n}{1 - r}$$

Therefore:

$$p + p \cdot (1-p)^1 + p \cdot (1-p)^2 + \cdots = \sum_{k=0}^{n-1} p(1-p)^k = p \cdot \frac{1 - (1-p)^n}{1 - (1-p)}$$

$$CDF(nT) = p \cdot \frac{1 - (1-p)^n}{1 - (1-p)} = 1 - (1-p)^n$$

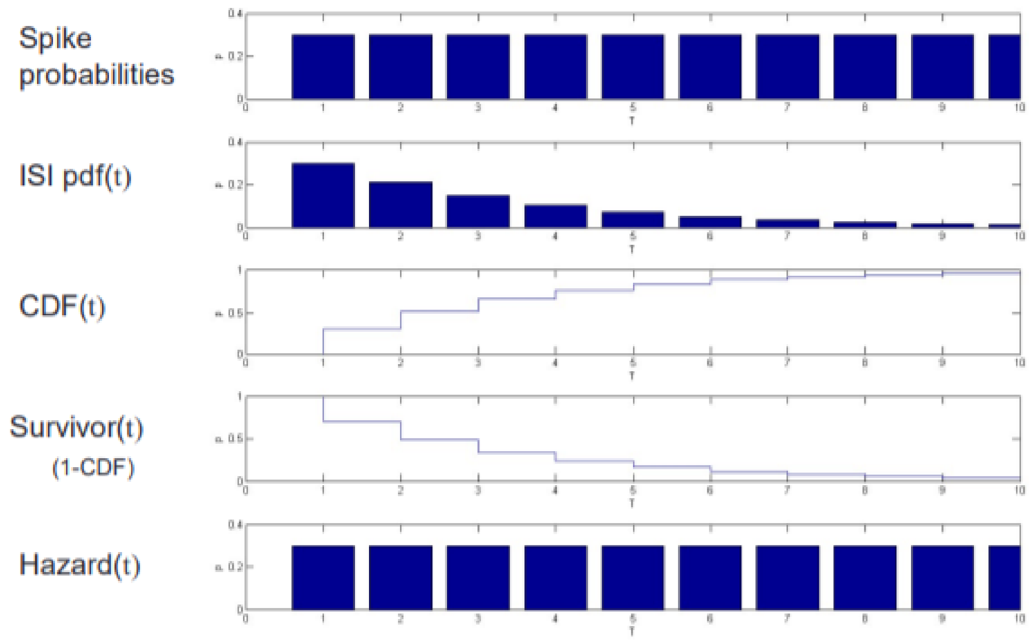
And the survivor function is:

$$Survivor(ISI = nT) = 1 - CDF(nT) = 1 - [1 - (1-p)^n] = (1-p)^n$$

To calculate the hazard function, we'll use the discrete form:

$$Hazard(ISI = nT) = \frac{P(ISI = nT)}{S(nT - \epsilon)} = \frac{(1-p)^{n-1} \cdot p}{(1-p)^{n-1}} = p$$

For example, when $p = 0.3$:



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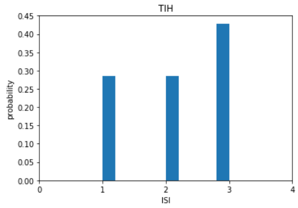
The Autocorrelation function:

If a regular neuron responds to two types of inputs, to the first one with firing rate every 3 ms and to the other with 5 ms, when the two inputs come together, its spike train will look like this:

100101100110100100101 (blue- spikes cause by input 1, green- input 2)

In this pattern, the TIH lack the information about the "regularity" of the process from input 2:



For this reason and others, many times we have more information about the system behavior when we compute the relations between spikes which are not adjacent. Autocorrelation is one of the relevant tools.

The Autocorrelation Definition:

For a discrete signal we use the definition:

$$Q(x,x)[n] = \sum_{i=-\infty}^{\infty} x[i] \cdot x[i+n]$$

Three Important notes:

- The autocorrelation is a function of **time lags**
- Autocorrelation can be defined also as
  $Q(x,x)[n] = Q(const\ reference, moving\ target)[number\ of\ shifting\ bins]$
- Can be used also with signals which are not spike trains by defining the result as the Pearson correlation.

Autocorrelation output could be either count, probabilities, or rates:

- Count: $Q_C(x,x)[n] = Q(x,x)[n] = \sum_{i=-\infty}^{\infty} x[i] \cdot x[i+n]$
- Probability: $Q_{prob}(x,x)[n] = \frac{1}{no.\ of\ spikes\ in\ x} Q_c(x,x)[n]$
- Rate: $Q_{rate}(x,y)[n] = \frac{1}{\Delta t} Q_{prob}(x,x)[n]$ $(\Delta t - bin\ size)$

Sanity checks for autocorrelation:

- Defining the max time lag (100-200 is commonly used in neuroscience)
- Symmetric
- We prefer to assign 0 at n=0 (no time lag)

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Visual explanation:

sample rate: 1000 Hz

Original: 0 1 1 0 0 1 0

Time lag = 1: 0 1 1 0 0 1 0

Time lag = 2: 0 1 1 0 0 1 0

Autocorrelation:

Time lag	0	1	2
Num. of correlated spikes	3	1	0
Prob. Of correlated spikes	1	1/3	0
Rate of correlated spikes	42	14	0

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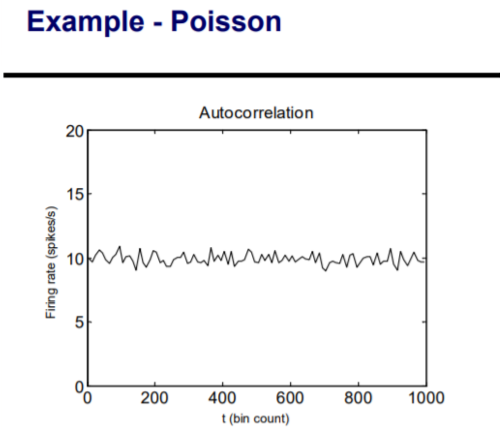
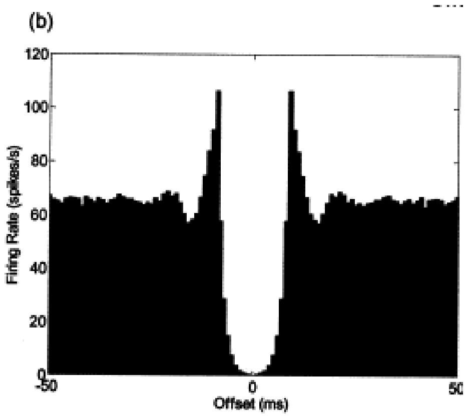
Class Exercise: Autocorrelation top examples

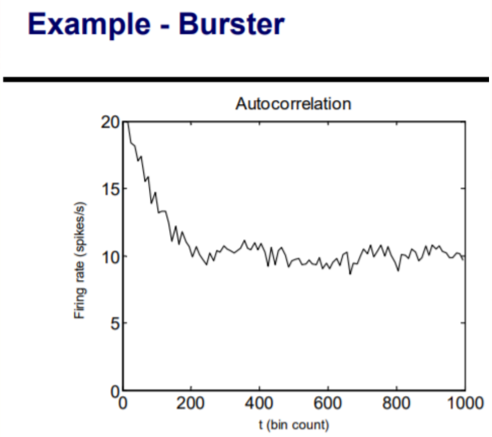
Sketch an autocorrelation of the following neurons given r=10 spikes/sec:

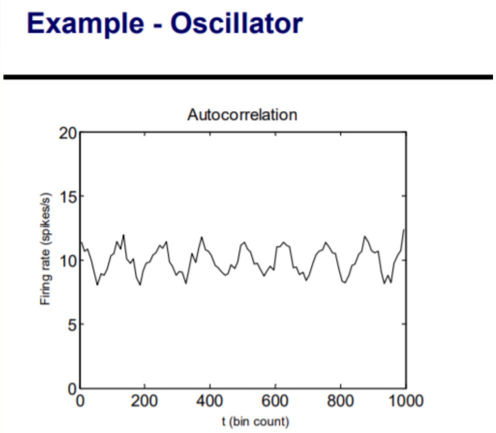
- Poisson neuron
- Poisson neuron with refractory period (lower rate function after a spike)
- Poisson burster (higher rate function after spikes)
- Oscillating neuron (Poisson with sinusoidal changing rates)

Solution:





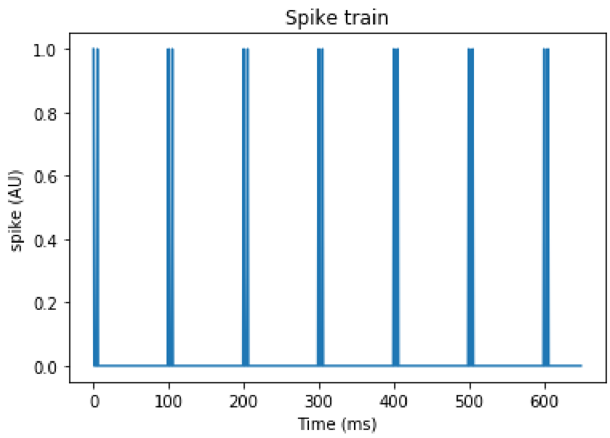




Examples from past exams:

Neurons in the Doubelitis Exactis nucleus of the Mountain Troll fire pairs of spikes (5 ms apart) every 100 ms.

- Calculate the Fano factor and Coefficient of variation of the neurons. Explain the results and their relation to a Poissonian neuron firing at the same rate.
- Draw the autocorrelation and hazard functions of the neurons (the functions should be drawn in the range 0-250msec).



Solution:

Reminder:

$$FF = \frac{Var(spaike\ count\ over\ T)}{E(spike\ count\ over\ T)}$$

$$CV = \frac{std(ISI)}{E(ISI)}$$

Fano Factor: We are free to choose any size of bin for spike count. The easiet will be 100 ms which will give us Var(no. of spikes over 100 ms)=0. Therefore Fano Factor is zero.

CV: We'll calculate-

$$P(ISI = 5) = P(ISI = 95) = 0.5$$

$$E(ISI) = 0.5 \cdot 5 + 0.5 \cdot 95 = 50$$

$$var(ISI) = 0.5 \cdot (5 - 50)^2 + 0.5 \cdot (95 - 50)^2 = 45^2$$

$$CV = \frac{std(ISI)}{E(ISI)} = \frac{50}{45^2} \to 0$$

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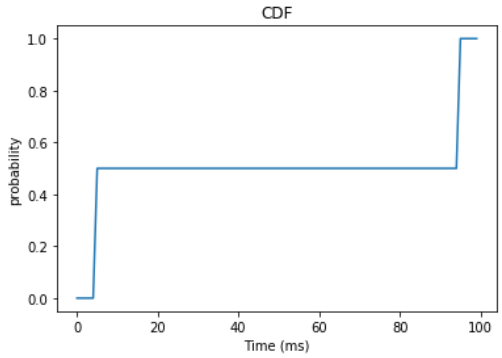
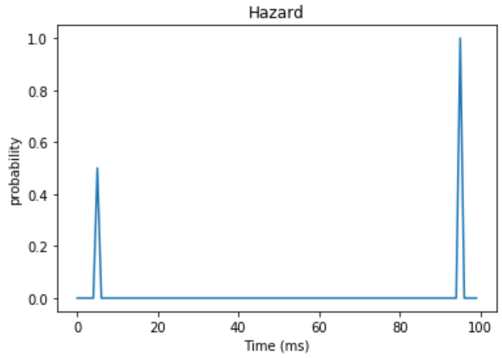
Hazard function:

$$Hazard(t) = \frac{PDF(ISI = t)}{1 - CDF(t - \epsilon)}$$





Autocorrelation function:

If we count possible time lags between spikes, we see that we have-

5 ms (for half of the spikes)

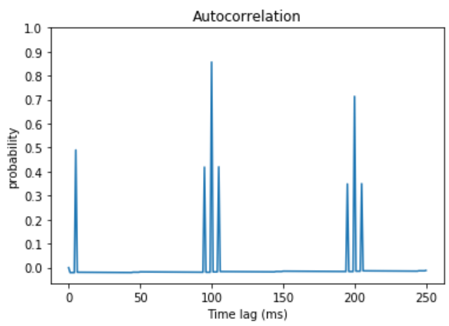
95 ms (for half of the spikes)

100 ms (for all spikes)

105 ms (for half of the spikes)

…

Therefore:



The Cross-correlation function:

Like autocorrelation, we can refer the “target” as a different neuron with different spike train, using the following definition (from Dayan and Abbot):

$$Q(x, y)[n] = \sum_{i=-\infty}^{\infty} x[i] \cdot y[i+n]$$

$$Q(x, y)[n] \equiv Q(const\ reference, moving\ target)[number\ of\ shifting\ bins]$$

Please note!

- Cross-correlation is a-symmetric
  - when using a python function, note how it defines the vectors. In some cases, it is defined as: $Q(x, y)[n] = \sum_{i=-\infty}^{\infty} x[i+n] \cdot y[i]$.
  - We won’t assign 0 for zero time-lag.
- The output of Cross-correlation, similarly to autocorrelation, can be either count, probability, or rate.

Class Exercise: Cross-Correlation of regular neurons

Neuron A fires regularly at 10 spikes/sec, t = 0, 100, 200, 300, 400, … ms.
Following each spike of A: neuron B fires 10 spikes at 1 ms intervals with initial delay of 20 ms. e.g: 20, 21, …29, 120, 121, etc.

- Draw example spike-trains of A and B
- Draw the cross correlation of A->B normalized to probability.
- Draw the cross correlation of B->A normalized to probability.
- Given the two cross-correlations, how can you decide if neuron A excites neuron B or vice versa?

Solution:

spike trains:

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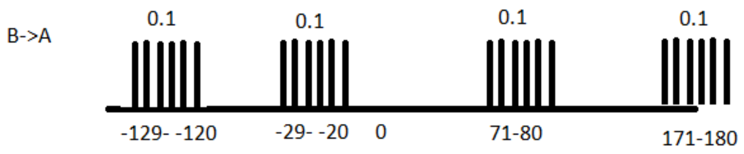
Cross correlation A->B, (probability 1):

Cross correlation B->A (probability 0.1):



The first cross-correlation implies that the probability is higher for firing spikes for neuron B after neuron A fired, and it begins already 20 ms after neuron A fired the spikes (in parallel to 70 ms when we look at the cross correlation from B to A). Therefore, early excitation time is more reasonable, and therefore neuron A excites neuron B.

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Class Exercise: Regular neuron excites Poisson neuron

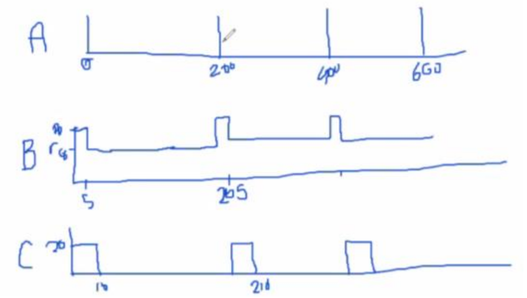
Neuron A sends excitatory inputs to neurons B and C (B and C are not directly connected).

- Neuron A fires regularly (a perfect pacemaker) at 5 spikes/sec
- Neuron B is a Poisson process with a rate of 60 spikes/s increasing to 80 spikes/s for a period of 5 ms following an incoming spike from neuron A.
- Neuron C is a Poisson process with a rate of 0 spikes/s increasing to 20 spikes/s for a period of 10ms following an incoming spike from neuron A.

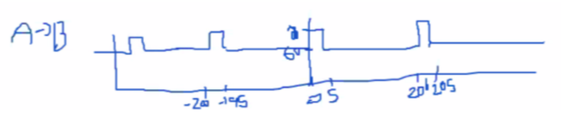
Draw the cross-correlation of A-B, A-C, B-C and C-B (in the range ±500ms).

Solution:

Illustration of the three neurons-



Cross correlation from A to B-



Similarly, cross correlation from A to C will look almost the same, but with highest of 0 and 20 and for periods of 10 ms.